Combined Transformations

Graphics 1 CMP-5010B

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- World Coordinate System
- Combined Transformations
- Articulated Structures

Combining 2D Transformations

Order matters!

Recap

So far we have looked at individual 2D transformations applied to the vertices of a 2D polygon.

Recap

Having put in place a uniform method, applicable to **all** transformations, we can now look at *combining* transformations.

Recap

We will show now that the *order* of the applied transformations is absolutely **crucial** to obtain the desired results!

World Coordinate System

Where in the world do we start?

World Coordinate System

Most graphics systems adopt a World Coordinate System (WCS), with a camera in a particular position and orientation.

World Coordinate System

For example in *OpenGL*, the camera is at the origin of the WCS pointing in the negative z-direction with its “up” vector pointing in the y-direction.

World Coordinate System

So far we have been working with 2D transformations.

Given the description of the OpenGL coordinate system, what has been the significance of the z-axis?

- Answer: It is the *axis* of **rotation**.

World Coordinate System

The WCS is represented by a *right-handed* coordinate system, with the z-axis popping out of the screen.

- For 2D, we draw in the x-y plane, and **rotate** about the z-axis.

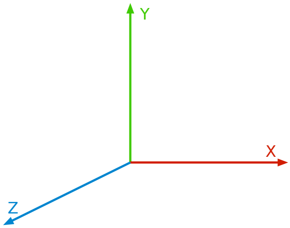

Figure 1: right handed coordinate system

Combined Transformations

Order matters. . .

Recap: Matrix Multiplication

Matrix multiplication **is** *associative*:

$$ABC = A(BC) = (AB)C$$

Matrix multiplication is **not** *commutative*:

$$AB \neq BA$$

Combined Transformations

Start with two common concatenated transformations:

1. *Rotate* the model, then *translate* it.
2. *Translate* the model, then *rotate* it.

Combined Transformations

Start with two common concatenated transformations:

1.
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2.
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Multiplying out the first example from right to left:

1.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} (x\cos\alpha - y\sin\alpha) + t_x \\ (x\sin\alpha + y\cos\alpha) + t_y \\ 1 \end{bmatrix}$$

Multiplying out the second example from right to left:

2.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} (x + t_x)\cos\alpha - (y + t_y)\sin\alpha \\ (x + t_x)\sin\alpha + (y + t_y)\cos\alpha \\ 1 \end{bmatrix}$$

Combined Transformations

$$RTv \neq TRv$$

The **order** of the transformations is important.

Articulated Structures

Hierarchical transformations...

Make the articulated structure from only the square polygon.

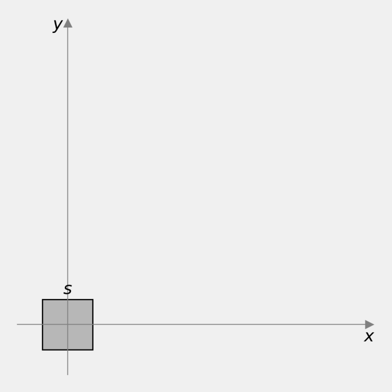

Figure 2: square, length = s

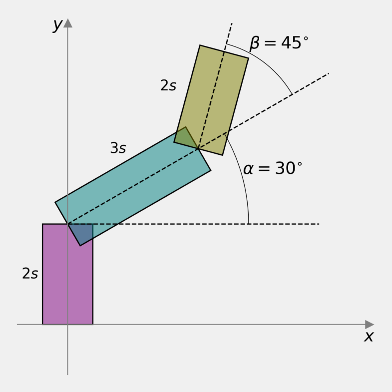

Figure 3: articulated structure

Assumptions

- The objects are in order of their position in the hierarchy, i.e. we start with the first object in the hierarchy, then the next and so on.
- The articulations should be operative rather than static, i.e for different value angles, the structure will still be connected.
- We only use the metric s in our transformation matrices.
- The solution may not be unique.

The base

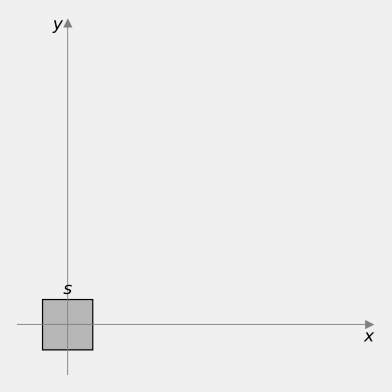

Figure 4: square, length $= s$

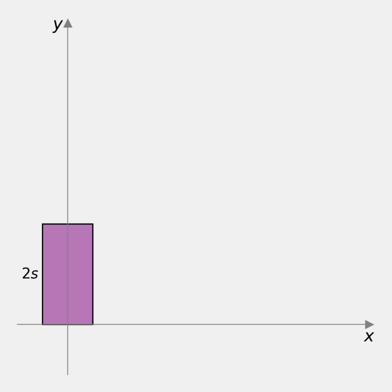

Figure 5: part 1, the base

The base

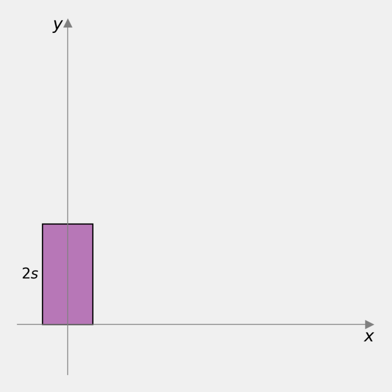

Figure 6: The base

Order of transformations:

1. scale in y, $s_y = 2$
2. translate in y, $t_y = 1s$

The base

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Link 1

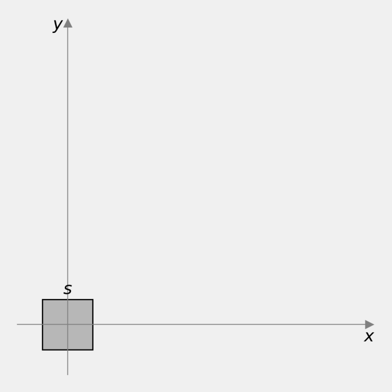

Figure 7: square, length $= s$

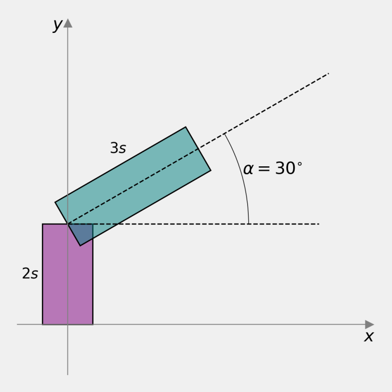

Figure 8: Link 1

Link 1

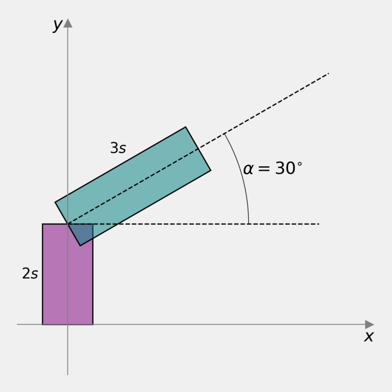

Figure 9: Link 1

Order of transformations:

1. scale in x, $s_x = 3$
2. translate in x, $t_x = 1.5s$
3. rotate, $\alpha = 30^\circ$
4. translate in y, $t_y = 2$

Link 1

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1.5s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Link 2

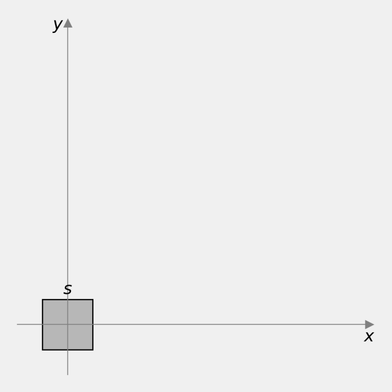

Figure 10: square, length $= s$

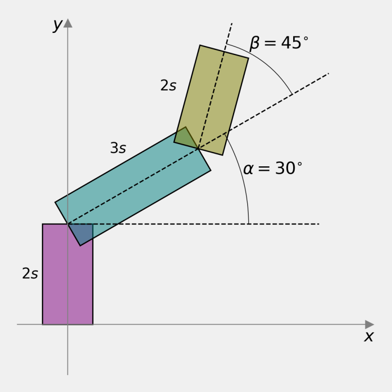

Figure 11: Link 2

Link 2

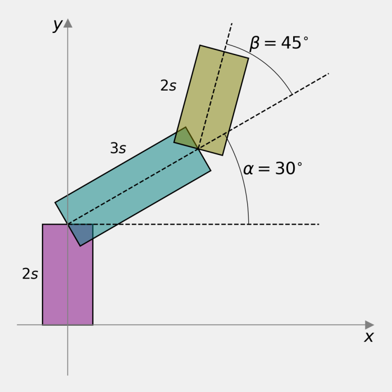

Figure 12: Link 2

Order of transformations:

1. scale in x, $s_x = 2$
2. translate in x, $t_x = s$
3. rotate, $\beta = 45^\circ$
4. translate in x, $t_x = 3s$
5. rotate, $\alpha = 30^\circ$
6. translate in y, $t_y = 2s$

Link 2

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NB: equation runs over two lines.

Articulated Arm Template

There is a template solution for each of the three parts.

Articulated Arm Template

Base:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Articulated Arm Template

Link 1:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ? & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\alpha = ?^\circ$$

Articulated Arm Template

Link 2

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ? & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\alpha = ?^\circ \ , \ \beta = ?^\circ$$

Summary

- World Coordinate System
- Combined Transformations
- Articulated Structures

Reading:

- Hearn, D. et al. (2004). Computer Graphics with OpenGL.
- Strang, Gilbert, et al. (1993) Introduction to linear algebra.